Engineering mathematics - IV

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Module-05

Sampling Theory and

Joint probability distributions

Joint probability and Joint probability distribution

If X and Y are two discrete random variables, we define the joint probability function of X and Y by

$$P(X = x, Y = y) = f(x, y)$$

where $f(x,y)$ satisfy the Conditions

$$f(x,y) \geq 0 \quad \text{and} \quad \sum_x \sum_y f(x,y) = 1$$

The Second Condition means that the sum of all values of x and y equal to one.

Joint probability table

X \ Y	y_1	y_2	...	y_n	Sum
x_1	J_{11}	J_{12}	...	J_{1n}	$f(x_1)$
x_2	J_{21}	J_{22}	...	J_{2n}	$f(x_2)$
...	...	...	...	...	...
x_m	J_{m1}	J_{m2}	...	J_{mn}	$f(x_m)$
Sum	$g(y_1)$	$g(y_2)$	...	$g(y_n)$	1

Marginal probability distribution

In the Joint probability table, $f(x_1)$, $f(x_2)$, ... $f(x_m)$ respectively represents the sum of all the entries in the first row, second row ... m^{th} row.
$g(y_1)$, $g(y_2)$, ... $g(y_n)$ respectively represents the sum of all the entries in the 1^{st} column, 2^{nd} ..... n^{th} column.

$$f(x_1) = J_{11} + J_{12} + \dots + J_{1n} \quad g(y_1) = J_{11} + J_{21} + \dots + J_{m1}$$
$$f(x_2) = J_{21} + J_{22} + \dots + J_{2n}, \quad g(y_2) = J_{12} + J_{22} + \dots + J_{m2}$$
$$\dots$$
$$f(x_m) = J_{m1} + J_{m2} + \dots + J_{mn}, \quad g(y_n) = J_{1n} + J_{2n} + \dots + J_{mn}$$

$\{f(x_1), f(x_2) \dots f(x_m)\}$ & $\{g(y_1), g(y_2) \dots g(y_n)\}$ are called marginal probability distribution of X & Y respectively.

NOTE: $f(x_1) + f(x_2) + \dots + f(x_m) = 1$
$g(y_1) + g(y_2) + \dots + g(y_n) = 1$

Expectation: Expectation of X is denoted by E(X) or μ_X

$$\mu_X = E(X) = \sum_{i=1}^{n} x_i f(x_i) \text{ or } \sum x f(x)$$

$$\mu_Y = E(Y) = \sum_{j=1}^{n} y_j g(y_j) \text{ or } \sum y g(y)$$

$$E(XY) = \sum x_i y_j J_{ij}$$

Covariance:-

Co-variance of X & Y is denoted by

$$\text{Cov}(X,Y) = E(XY) - \mu_X \mu_Y$$

$+J_{m1}$

$+J_{m2}$

..

$+J_{mn}$

Correlation:-

Correlation of X & Y is denoted by

~~Cov (X (Y) et E((XY)~~

Not in Sylabus

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

$$\sigma_X^2 = E(X^2) - \mu_X^2$$

$$\sigma_Y^2 = E(Y^2) - \mu_Y^2$$

$$E(X^2) = \sum x_i^2 f(x_i)$$

$$E(Y^2) = \sum y_j^2 g(y_j)$$

NOTE: If X & Y are independent random variable then ① $E(XY) = E(X) \cdot E(Y)$

② $\text{Cov}(X,Y) = 0$ & hence $\rho(X,Y) = 0$

Problems

1. The joint distribution of two random variables X & Y is as follows.

X \ Y	-4	2	7
1	1/8	1/4	1/8
5	1/4	1/8	1/8

Compute
1. E(X) and E(Y)
2. E(XY)
3. Cov(X, Y)
4. σ_x^2, σ_y

Soln:

X \ Y	-4	2	7	sum
1	1/8	1/4	1/8	1/2
5	1/4	1/8	1/8	1/2
sum	3/8	3/8	1/4	1

$\}f(x)$ (row sums); $g(y)$ (column sums)

1. $E(X) = \sum x_i f(x_i)$

$= 1 \times 1/2 + 5 \times 1/2$

$= 1/2 + 5/2$

$= 6/2$

$E(X) = 3$

2. $E(Y) = \sum y_j g(y_j)$

$= -4 \times \frac{3}{8} + 2 \times \frac{3}{8} + 7 \times 1/4$

$= -\frac{12}{8} + \frac{6}{8} + \frac{7}{4}$

$= \frac{-12 + 6 + 14}{8}$

$= \frac{8}{8} = 1 \Rightarrow E(Y) = 1$

E(X

They $\mu_x = E(X) = 3$

$\mu_y = E(Y) = 1$

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② $E(XY) = \sum x_i y_j J_{ij}$

$= (1)(-4)(1/8) + (1)(2)(1/4) + (1)(7)(1/8)$
$+ 5(-4)(1/4) + 5(2)(1/8) + 5(7)(1/8)$

$= -1/2 + 1/2 + 7/8 - 5 + 5/4 + 35/8$

$= \frac{42}{8} - \frac{15}{4}$

$= \frac{42-30}{8}$

$= \frac{12}{8}$

$E(XY) = \frac{3}{2}$ //

③ $\text{cov}(X,Y) = E(XY) - \mu_x \mu_y$

$= \frac{3}{2} - (3)(1)$

$7 \times 1/4$

$= \frac{3-6}{2}$

$= -3/2$ //

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(ii) $\sigma_x^2 = E(x^2) - \mu_x^2$

$$E(x^2) = \sum x_i^2 f(x_i)$$

$= (1)^2 (1/2) + 5^2 (1/2)$

$= \frac{1}{2} + \frac{25}{2}$

$= 26/2$

$E(x^2) = 13$

$\sigma_x^2 = 13 - (3)^2 = 13 - 9 = 4$

$\sigma_x = \sqrt{4} \Rightarrow \sigma_x = 2$

$\sigma_y^2 = E(y^2) - \mu_y^2$

$E(y^2) = \sum y_j^2 \cdot g(y_j)$

$E(y^2) = (-4)^2 (3/8) + 2^2 (3/8) + 7^2 (1/4)$

$= 16 \times 3/8 + 4 \times 3/8 + 49 \times 1/4$

$= \frac{48 + 12 + 98}{8}$

$= \frac{\cancel{158}\ 79}{\cancel{8}\ 4}$

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$E(Y^2) = \frac{79}{4}$

$\sigma^2_y = \frac{79}{4} - (1)^2 = \frac{75}{4}$

$\sigma_y = \sqrt{75/4} = 4.33$ //

② The Joint probability distribution table for two random variables X and Y is as follows.

X \ Y	-2	-1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

Determine the marginal probability distribution of X and Y also Compute

① Expectation of X, Y and XY

② S. Ds of X, Y

③ Covariance of X and Y

further Verify that X and Y are dependent random variables

Soln:

X \ Y	-2	-1	4	5	sum
1	0.1	0.2	0	0.3	0.6
2	0.2	0.1	0.1	0	0.4
Sum	0.3	0.3	0.1	0.3	1

last column & last row are marginal probability distributions.

① $E(X) = \sum x_i f(x_i)$

$= (1)(0.6) + 2(0.4)$

$= 0.6 + 0.8$

$E(X) = 1.4$

$E(X) = \mu_X = 1.4$

$E(Y) = \sum y_j g(y_j)$

$= -2(0.3) + (-1)(0.3) + 4(0.1) + 5(0.3)$

$E(Y) = 1$

$E(Y) = \mu_Y = 1$

$E(XY) = \sum x_i y_j J_{ij}$

$= (1)(-2)(0.1) + (1)(-1)(0.2) + (1)(4)(0) + (1)(5)(0.3) + 2(-2)(0.2) + 2(-1)(0.1) + 2(4)(0.1) + 2(5)(0)$

$= 0.9$

② $\sigma^2_X = E(X^2) - \mu^2_X$ & $\sigma^2_Y = E(Y^2) - \mu^2_Y$

$E(X^2) = \sum x_i^2 f(x_i)$

$= (1)^2(0.6) + 2^2(0.4)$

$= 0.6 + 1.6$

$E(X^2) = 2.2$

$\sigma^2_X = E(X^2) - \mu^2_X$

$= 2.2 - (1.4)^2$

$= 0.24$

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$\sigma_x = \sqrt{0.24}$

$\sigma_x = \underline{0.489}$

$E(Y^2) = \sum y_j^2\, g(y_j)$

$= (-2)^2(0.3) + (-1)^2(0.3) + 4^2(0.1) + 5^2(0.3)$

$= 4(0.3) + 0.3 + 16(0.1) + 25(0.3)$

$E(Y^2) = 10.6$

$\sigma^2 y = E(Y^2) - \mu^2 y$

$\sigma^2 y = 10.6 - 1^2$

$= 9.6$

$\sigma_y = \sqrt{9.6} = \underline{3.09}$

$\sigma_x = \underline{0.489}$ $\sigma_y = 3.09$ //

(3) $cov(X, Y) = E(XY) - E(X)E(Y)$

$= 0.9 - (1.4)(1)$

$= \underline{-0.5}$

If X & Y are independent random variables we must have $f(x_i)\, g(y_j) = J_{ij}$

Now $f(x_1)g(y_1) = (0.6)(0.3) = 0.18$

but $J_{11} = 0.1$

$0.18 \neq 0.1$

i.e $f(x_1)\,g(y_1) \neq J_{11}$

III'y for other valuy alyo condition is not satisfied.

hence we conclude that X & Y are dependent random variables.

(3) Do yourself

The joint probability distribution table for two random variables X and Y is as follows

X \ Y	-2	5	8	
1	0.21	0.35	0.14	
2	0.09	0.15	0.06	

determine the marginal probability distribution of X and Y, alyo find

① E(XY), E(X), E(Y)

② COV(X, Y)

(4) Suppose X and Y are independent random variably with the following respective distribution, find the Joint distribution of X and Y. Also verify that $cov(X,Y)=0$

x_i	1	2
$f(x_i)$	0.7	0.3

y_j	-2	5	8
$g(y_j)$	0.3	0.5	0.2

Solⁿ:

X \ Y	$y_1=-2$	$y_2=5$	$y_3=8$	$f(x_i)$/sum
$x_1=1$	J_{11}	J_{12}	J_{13}	0.7
$x_2=2$	J_{21}	J_{22}	J_{23}	0.3
$g(y_j)$/sum	0.3	0.5	0.2	1

J_{ij} are obtained on multiplication of marginal entries

$J_{11}=(0.3)(0.7)=0.21$

$J_{12}=(0.7)(0.5)=0.35$

$J_{13}=(0.7)(0.2)=0.14$

$J_{21}=(0.3)(0.3)=0.09$

$J_{22}=(0.5)(0.3)=0.15$

$J_{23}=(0.3)(0.2)=0.06$

Joint distribution table is

X \ Y	-2	5	8	$f(x_i)$/sum
1	0.21	0.35	0.14	0.7
2	0.09	0.15	0.06	0.3
$g(y_j)$ or sum	0.3	0.5	0.2	1

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$$\mathrm{COV}(X,Y) = E(XY) - \mu_X \mu_Y$$

$$\mu_X = E(X) = \sum x_i f(x_i)$$

$$= (1)(0.7) + 2(0.3)$$

$$= 1.3$$

$$\mu_Y = E(Y) = \sum y_j g(y_j)$$

$$= (-2)(0.3) + (5)(0.5) + 8(0.2)$$

$$= 3.5$$

$$E(XY) = \sum x_i y_j J_{ij}$$

$$= (1)(-2)(0.21) + (1)(5)(0.35) + (1)(8)(0.14) + (2)(-2)(0.09) + (2)(5)(0.15) + (2)(8)(0.06)$$

$$= -0.42 + 1.75 + 1.12 - 0.36 + 1.5 + 0.96$$

$$= 4.55$$

$$\mathrm{COV}(X,Y) = 4.55 - (1.3)(3.5)$$

$$= 4.55 - 4.55$$

$$= 0$$

(5) X and Y are independent variables X takes value 2, 5, 7 with probability $1/2$, $1/4$, $1/4$ respectively. Y takes values 3, 4, 5 with the probability $1/3, 1/3, 1/3$

(a) find the joint probability distribution of X and Y.

(b) S.T the covariance of X and Y is equal to zero

(c) find the probability distribution of $Z = X + Y$

Sol^n:

X \ Y	3	4	5	$f(x_i)$
2	J_{11}	J_{12}	J_{13}	$1/2$
5	J_{21}	J_{22}	J_{23}	$1/4$
7	J_{31}	J_{32}	J_{33}	$1/4$
$g(y_j)$	$1/3$	$1/3$	$1/3$	1

$J_{ij} = f(x_i)\, g(y_j)$

$J_{11} = (1/3)(1/2) = 1/6$ $\quad J_{21} = 1/3\,(1/4) = 1/12$

$J_{12} = (1/3)(1/2) = 1/6$ $\quad J_{22} = 1/3\,(1/4) = 1/12$

$J_{13} = (1/3)(1/2) = 1/6$ $\quad J_{23} = 1/3\,(1/4) = 1/12$

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$J_{31} = 1/12, \; J_{32} = 1/12, \; J_{33} = 1/12$

X \ Y	3	4	5	$f(x_i)$
2	1/6	1/6	1/6	1/2
5	1/12	1/12	1/12	1/4
7	1/12	1/12	1/12	1/4
$g(y_j)$	1/3	1/3	1/3	1

(b) $cov(XY) = E(XY) - \mu_X \mu_Y$

$\mu_X = E(X) = \sum x_i f(x_i)$

$= 2(1/2) + 5(1/4) + 7(1/4)$

$= 1 + 5/4 + 7/4$

$= 1 + \frac{12}{4}$

$= 16/4$

$\mu_X = 4$

$\mu_Y = E(Y) = \sum y_j g(y_j)$

$\mu_Y = (1/3)(3) + 4(1/3) + 5(1/3)$

$= \frac{3+4+5}{3}$

$= 12/3 = 4$

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$E(XY) = \sum x_i y_i J_{ij}$

$= (2)(3)(1/6) + (2)(4)(1/6) + (2)(5)(1/6) + 5(3)(1/12) + 5(4)(1/12) + (5)(5)(1/12) + 7(3)(1/12) + 7(4)(1/12) + 7(5)(1/12)$

$= 16$

$cov(X,Y) = 16 - 4(4) = 16 - 16 = 0$

(2) $Z = X + Y$

Let $Z_i = x_i + y_i$

$Z_i = \{5, 6, 7, 8, 9, 10, \cancel{10}, 11, 12\}$

corresponding probabilities are

$1/6, 1/6, 1/6, 1/12, 1/12, 1/12, 1/12, 1/12, 1/12, 1/12$

probability distribution of $Z = X + Y$ is as follows

Z	5	6	7	8	9	10	11	12
P(Z)	1/6	1/6	1/6	1/12	1/12	$1/12 + 1/12 = 1/6$	1/12	1/12

$5+5=10$
$7+3=10$
for both $1/12 + 1/12 = 2/12 = 1/6$

$\sum P(Z) = 3/6 + \frac{2}{12} + 1/6 + \frac{2}{12}$

$= \frac{3+1+1+1}{6}$

$= 6/6$

$= 1$

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(*) Given the following joint distribution of the random variables X & Y, find the corresponding marginal distribution also compute covariance & correlation of random variable X & Y.

X \ Y	1	3	9
2	1/8	1/24	1/12
4	1/4	1/4	0
6	1/8	1/24	1/12

Soln: marginal distribution of X & Y is

x_i	2	4	6
$f(x_i)$	1/4	1/2	1/4

y_j	1	3	9
$g(y_j)$	1/2	1/3	1/6

For understanding

$f(x_1) = 1/8 + 1/24 + 1/12 = 12/48 = 1/4$

$f(x_2) = 1/4 + 1/4 + 0 = 2/4 = 1/2$

$f(x_3) = 1/8 + 1/24 + 1/12 = 1/4$

$g(y_1) = 1/8 + 1/4 + 1/8 = 2/8 + 1/4 = 2/4 = 1/2$

$g(y_2) = 1/24 + 1/4 + 1/24 = 2/24 + 1/4 = 1/12 + 1/4 = \frac{1+3}{12} = 1/3$

$g(y_3) = 1/12 + 0 + 1/12 = \frac{2}{12} = 1/6$

$COV(X,Y) = E(XY) - \mu_x \mu_y$

$E(X) = \mu_x = \sum x_i f(x_i)$

$= (2)(1/4) + (4)(1/2) + (6)(1/4)$

$= 4$

$E(Y) = \mu_y = \sum y_j g(y_j)$

$= (1)(1/2) + 3(1/3) + 9(1/6)$

$= 3$

$E(XY) = \sum x_i y_j J_{ij}$

$= (2)(1)(1/8) + 2(3)(1/24) + 2(9)(1/12)$

$+ (4)(1)(1/4) + (4)(3)(1/4) + (4)(9)(0) + (6)(1)(1/8)$

$+ (6)(3)(1/24) + (6)(9)(1/12)$

$E(XY) = 12$

$Cov(X,Y) = 12 - 4(3)$

$= 0$

(7) The Joint probability distribution of two discrete random variables X & Y is given by $f(x,y) = K(2x+y)$ where x & y are integers such that

$0 \le x \le 2,\ 0 \le y \le 3$

$= \frac{1+3}{12} = 1/3$

(a) find K

(b) find the marginal distributions of X & Y

(c) S.T the random variables X & Y are dependent

Sol^n: $X = \{x_i\} = \{0, 1, 2\}$

$Y = \{y_j\} = \{0, 1, 2, 3\}$

$f(x, y) = K(2x + y)$ Joint probability distribution table is

X \ Y	0	1	2	3	sum
0	0	K	2K	3K	6K
1	2K	3K	4K	5K	14K
2	4K	5K	6K	7K	22K
Sum	6K	9K	12K	15K	42K

(a) we must have $42K = 1$

$K = 1/42$

(b) marginal probability distribution is

x_i	0	1	2
$f(x_i)$	$6/42 = 1/7$	$14/42 = 1/3$	$22/42 = 11/21$

y_j	0	1	2	3
$g(y_j)$	$6/42 = 1/7$	$9/42 = 3/14$	$12/42 = 2/7$	$15/42 = 5/14$

(c) It can be easily seen that $f(x_i)\,g(y_j)\,f_{ij}$ hence random variables are dependent.

Stochastic process

Definitions :-

probability vector :

The vector $V = (v_1, v_2, v_3, \dots v_n)$ is called a probability vector if each one of its components are non negative & their sum is equal to unity.

ex: $V = (1, 0)$

$V = (1/2, 1/2)$

$V = (1/4, 1/4, 1/2)$

are all probability vector

Stochastic matrix :- A square matrix P is said to be stochastic matrix if every row in the form of a probability vector.

exc: ① $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ② $\begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}$

Regular Stochastic matrix

A stochastic matrix P is said to regular stochastic matrix if all the entries of some power p^n are +ve ex : $A = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$

Consider $A^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 3/4 \end{bmatrix}$

A is a regular stochastic matrix

Properties of Regular Stochastic matrix

① ⓐ P has a unique fixed point $x = (x_1, x_2 \ldots x_n) \ni; xP = x$

ⓑ P has unique fixed probability vector $V = (V_1, V_2 \ldots V_n) \ni; VP = V$

where $V_i = \frac{x_i}{\sum_{i=1}^{n} x_i}$

ⓒ $P^2, P^3 \ldots$ approches the matrix V whose rows are fixed probability vector V.

ⓓ If u is any probability vector then the sequence of vectors $uP, uP^2 \ldots uP^n$ approches the unique fixed probability vector V.

Problems :

① If $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is a Stochastic matrix and $V = [v_1, v_2]$ is a probability vector S.T VA is also a probability vector.

Soln: By data

$$a_1 + a_2 = 1, \quad b_1 + b_2 = 1, \quad v_1 + v_2 = 1$$

$$VA = [v_1 \; v_2] \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

$$VA = \begin{bmatrix} v_1 a_1 + v_2 b_1 & v_1 a_2 + v_2 b_2 \end{bmatrix}$$

we have to probability vector

So, $v_1 a_1 + v_2 b_1 + v_1 a_2 + v_2 b_2 = 1$

$$v_1 (a_1 + a_2) + v_2 (b_1 + b_2) = 1$$

$$v_1 (1) + v_2 (1) = 1$$

$$v_1 + v_2 = 1$$

$\therefore$ VA is a probability vector

② Find the unique fixed probability vector of the regular Stochastic matrix $A = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$

Sol^n: we have to find $v=(x,y)$ where

$x+y=1$ $\ni$; $vA=v$

$$[x, y]\begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = [x, y]$$

$$\left[\frac{3}{4}x+\frac{y}{2}, \frac{x}{4}+\frac{y}{2}\right] = [x, y]$$

$$\frac{3}{4}x+\frac{y}{2} = x \quad -① $$

$$\frac{x}{4}+\frac{y}{2} = y \quad -②$$

w.r.t $x+y=1$

$y=1-x$

put $y=1-x$ in ①

$$\frac{3}{4}x+\frac{1-x}{2} = x$$

$$3x+2(1-x) = 4x$$

$$3x+2-2x = 4x$$

$$3x-2x-4x+2=0$$

$$-3x+2=0$$

$$-3x=-2$$

$$x=2/3$$

$y = 1 - x = 1 - \frac{2}{3}$

$y = \frac{1}{3}$, $V = (x, y) = (2/3, 1/3)$

Thus $(2/3, 1/3)$ is the unique fixed probability vector.

(3) Find the unique fixed probability vector for the regular stochastic matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$$

Soln: we have to find $V = (x, y, z)$ where $x + y + z = 1$ $\ni$; $VA = V$

$$[x, y, z] \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [x, y, z]$$

$$[0 + y/6 + 0,\ x + y/2 + 2/3 z,\ 0 + y/3 + z/3] = [x, y, z]$$

$$y/6 = x, \quad x + y/2 + 2/3 z = y, \quad y/3 + z/3 = z$$

$$y = 6x, \quad 6x + 3y + 4z = 6y, \quad y + z = 3z$$

$$y = 6x, \quad 6x - 3y + 4z = 0, \quad y - 2z = 0$$

$-(1)$ $\qquad -(2)$ $\qquad -(3)$

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w.k.T $x+y+z=1$

$z = 1-x-y$

w.k.T $y = 6x$

$z = 1-x-6x$

$z = 1-7x$

③ becomes

$y - 2z = 0$

$y - 2(1-7x) = 0$

$6x - 2 + 14x = 0$

$20x - 2 = 0$

$20x = 2$

$x = 1/10$

$y = 6x = 6 \times 1/10$

$y = 3/5$, $z = 1 - 7x$

$= 1 - 7 \times 1/10$

$= 1 - 7/10$

$z = \frac{3}{10}$

required unique fixed probability vector v is given by $v = (1/10, 3/5, 3/10)$

(4) S.T $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is a regular stochastic matrix. Also find the associated unique fixed probability vector

Solⁿ: consider $P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

$$P^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^3 = P \cdot P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$P^4 = P \cdot P^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$P^5 = P \cdot P^4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$$

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we observe that P^5 all the entries are +ve.

Thus P is a regular stochastic matrix

we have to find $v = (a, b, c)$ where $a+b+c=1$ ∋; $vP = v$

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [a, b, c]$$

$$\left[0+0+\frac{c}{2},\ a+0+\frac{c}{2},\ 0+b+0\right] = [a, b, c]$$

$$\left[\frac{c}{2},\ a+\frac{c}{2},\ b\right] = [a, b, c]$$

$$\frac{c}{2} = a, \quad a+\frac{c}{2} = b, \quad b = c$$

$$c = 2a, \quad a+\frac{c}{2} = b, \quad b = c$$
$$-(1) \qquad -(2) \qquad -(3)$$

$$b = c$$
$$b = 2a$$

use $b = 2a$ in (2)

$$a + \frac{c}{2} = 2a$$

$$\frac{c}{2} = a$$

$$c = 2a$$

w.k.t $a+b+c=1$

$a + 2a + 2a = 1$

$5a = 1$

$a = \frac{1}{5}$

$\therefore \; b = \frac{2}{5}, \quad c = \frac{2}{5}$

Thus $(1/5, 2/5, 2/5)$ is the required unique fixed probability vector of P.

(5) Find the unique fixed probability vector of the regular stochastic matrix

$$P = \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$

Solⁿ: we have to find $V = (a, b, c, d)$ where $a+b+c+d = 1 \ni; \; VP = V$

$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/4 & 1/4 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b & c & d \end{bmatrix}$$

$$\left[0 + \frac{b}{2} + \frac{c}{2} + \frac{d}{2}, \; \frac{a}{2} + 0 + \frac{c}{2} + \frac{d}{2}, \; \frac{a}{4} + \frac{b}{4} + 0 + 0, \; a/4 + b/4 + 0 + 0 \right] = \begin{bmatrix} a & b & c & d \end{bmatrix}$$

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$$\left[\frac{b}{2}+\frac{c}{2}+\frac{d}{2},\ \frac{a}{2}+\frac{c}{2}+\frac{d}{2},\ \frac{a}{4}+\frac{b}{4},\ \frac{a}{4}+\frac{b}{4}\right]=[a,\ b,\ c,\ d]$$

$\frac{b}{2}+\frac{c}{2}+\frac{d}{2}=a \Rightarrow b+c+d=2a$ — ①

$\frac{a}{2}+\frac{c}{2}+\frac{d}{2}=b \Rightarrow a+c+d=2b$ — ②

$\frac{a}{4}+\frac{b}{4}=c \Rightarrow a+b=4c$ — ③

$\frac{a}{4}+\frac{b}{4}=d \Rightarrow a+b=4d$ — ④

w.K.T $a+b+c+d=1$

$\underline{b+c+d}=1-a$

but use ① $2a=1-a$

$3a=1 \Rightarrow \underline{\underline{a=1/3}}$

$a+b+c+d=1$

use ② $\underline{a+c+d}=1-b$

$\hookrightarrow 2b=1-b$

$3b=1$

$\underline{\underline{b=1/3}}$

③ $\Rightarrow a+b=4c$

$1/3+1/3=4c$

$2/3=4c$

$c=\frac{2}{3\times4} \Rightarrow \underline{\underline{c=1/6}}$

or countab,

b, c, d]

$a+b=4d$

$\frac{1}{3}+\frac{1}{3}=4d$

$\frac{2}{3}=4d$

$d=\frac{1}{6}$

Thus $V=(\frac{1}{3},\frac{1}{3},\frac{1}{6},\frac{1}{6})$ is the required unique fixed probability vector

Markov chains

A stochastic process which is ∋; the generation of the probability distribution depend only on the present state is called Markov process.

If the state space is discrete (finite or countably infinite) we say that the process is a discrete state process or chain. Then the markov process is known as a Markov chain.

It is defined as

① Each outcome belong to the finite set of the outcomes $\{a_1, a_2, \dots a_m\}$

(ii) The outcome of any trial depend at most upon the outcome of the immediate preceeding trial.
probability P_{ij} is associated with every pair of sates (a_i, a_j) that a_j occurs immediatly after a_i occurs. Such a stochastic process is called a finite markov chain

Transistion probability matrix
TPM of order m is denoted by P

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & & p_{mm} \end{bmatrix}$$

elements of P have the following properties

① $0 \leq P_{ij} \leq 1$

② $\sum_{j=1}^{m} P_{ij} = 1$

$(i = 1, 2, \ldots m)$

The above 2 propertiy satisfy the requirement of a stochastic matrix and hence we conclude that transition matrix of a markov chain is a stochastic matrix.

Higher transition probabilities

* The entry P_{ij} in the transition probability matrix P of the markov chain is the probability that system changes from state a_i to a_j in a single step i.e $a_i \to a_j$
* The probability that system changes from state a_i to a_j in exactly n steps is denoted by $P_{ij}^{(n)}$
* Let $p^{(n)} = [P_1^{(n)}, P_2^{(n)} \ldots P_m^{(n)}]$ denote the n^{th} step probability distribution at the end of n steps. Thus we have

$$p^{(1)} = p^{(0)} P, \quad p^{(2)} = p^{(1)} P = p^{(0)} . P . P = p^{(0)} P^2$$
$$\ldots \quad p^{(n)} = p^{(0)} P^n$$

problems

① The transition matrix P of a markov chain is given by $\begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}$ with the initial probability distribution $p^{(0)} = (1/4, 3/4)$

Define and find the following

① $P_{21}^{(2)}$ ② $P_{12}^{(2)}$ ③ $P_1^{(2)}$

② the vector $p^{(0)} p^{(n)}$ approaches

③ the matrix P^n approaches

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Sol'; First we need to find P^2

$$P^2 = P \cdot P = \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 5/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix}$$

(i) $P_{21}^{(2)} = 9/16$ (ii) $P_{12}^{(2)} = 3/8$

(iii) $p^{(2)} = p^{(0)} P^{(2)}$

$$= [1/4, 3/4] \begin{bmatrix} 5/8 & 3/8 \\ 9/16 & 7/16 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{37}{64}, \frac{27}{64} \end{bmatrix} = \begin{bmatrix} p_1^{(2)}, p_2^{(2)} \end{bmatrix}$$

(iv) $p_1^{(2)} = \frac{37}{64}$

i.e $p_1^{(2)}$ is the probability that the process is in the state a_1 after 2 steps

(v) The vector $p^{(0)} P^n$ approches the unique fixed probability vector P

let $V = (x, y)$ where $x + y = 1$

$$VP = V$$

$$[x, y]\begin{bmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{bmatrix} = [x, y]$$

$$\left[\frac{x}{2} + \frac{3y}{4}, \frac{x}{2} + \frac{y}{4}\right] = [x, y]$$

$$\frac{x}{2} + \frac{3y}{4} = x$$

$$2x + 3y = 4x$$

$$3y = 2x$$

but $x + y = 1$

$$y = 1 - x$$

$$3(1-x) = 2x$$

$$3 - 3x = 2x$$

$$3 = 5x$$

$$x = 3/5$$

$$x + y = 1 \Rightarrow y = 1 - x = 1 - 3/5 = 2/5$$

the vector $p^{(0)} p^n$ approches the vector $(3/5, 2/5)$

(vi) p^n approches the matrix $\begin{bmatrix} 3/5 & 2/5 \\ 3/5 & 2/5 \end{bmatrix}$

p^n approches the matrix V whose rows are each the fixed probability vector p.

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(2) The T.P.M of a markov chain is given by $P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$

and the initial probability distribution is $p^{(0)} = (1/2, 1/2, 0)$

Find $P_{13}^{(2)}$, $P_{23}^{(2)}$, $p^{(2)}$ and $P_1^{(2)}$

Sol^n: Let us find two step transition matrix P^2

$$P^2 = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} = \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{bmatrix}$$

$P_{13}^{(2)} = 3/8$ and $P_{23}^{(2)} = 1/2$

$$p^{(2)} = p^{(0)} P^{(2)} = \left[1/2, 1/2, 0\right] \begin{bmatrix} 3/8 & 1/4 & 3/8 \\ 1/2 & 0 & 1/2 \\ 11/16 & 1/8 & 3/16 \end{bmatrix}$$

$$p^{(2)} = \left[7/16, 1/8, 7/16\right]$$

$$P_1^{(2)} = 7/16$$

(3) P.T the markov chain whose t.p.m is $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is irreducible. Find the corresponding stationary probability vector

Soln: we shall S.T P is a regular stochastic matrix

p^2

$$P^2 = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/4 & 7/12 & 1/6 \\ 1/4 & 1/3 & 7/12 \end{bmatrix}$$

Since all the entry P^2 are +ve we conclude that t.p.m P is regular hence it is irreducible.

we shall find the fixed probability vector of P.

If $V = (x, y, z)$ we shall find V ∋;

$VP = V$ where $x+y+z=1$

$$[x, y, z] \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [x \; y \; z]$$

$$\left[\frac{y}{2}+\frac{z}{2},\ \frac{2}{3}x+\frac{z}{2},\ \frac{x}{3}+\frac{y}{2}\right]=\left[x\ \ y\ \ z\right]$$

$$\frac{y}{2}+\frac{z}{2}=x$$

$$2x=y+z \Rightarrow y+z=2x \Rightarrow 2x-y-z=$$

$$\frac{2}{3}x+\frac{z}{2}=y \Rightarrow 4x+3z=6y \Rightarrow 4x-6y+$$

$$\frac{x}{3}+\frac{y}{2}=z \Rightarrow 2x+3y=6z$$

$$2x+3y-6z=0 \quad -(3)$$

w.k.t $x+y+z=1$

$$x=1-y-z$$

(1) => [illegible]

[illegible]

(2) => [illegible]

$$(1) \Rightarrow 2(1-y-z)-y-z=0$$

$$2-2y-2z-y-z=0$$

$$2-3y-3z=0$$

$$-3y-3z+2=0$$

$$3y+3z-2=0 \quad -(4)$$

$$② \Rightarrow 4(1-y-z) - 6y + 3z = 0$$

$$4 - 4y - 4z - 6y + 3z = 0$$

$$4 - 10y - z = 0$$

$$-10y - z + 4 = 0 = 0 \quad - ⑤$$

$0 - ①$

$z = 0 - ②$

Solve ④ & ⑤

$$3y + 3z - 2 = 0$$

$$(-)\,4y \pm z \mp 8 = 0$$

$$-y + 0 = 0$$

$$\underline{y = 0}$$

Solve ① & ②

$$2x - y - z = 0$$

$$(-)\,2x \pm 3y \mp 6z = 0$$

$$-4y + 5z = 0 \quad - ⑤$$

Solve ④ & ⑤

$$3y + 3z - 2 = 0 \times 4$$

$$-4y + 5z = 0 \times 3$$

$$12y + 12z - 8 = 0$$

$$-12y + 15z = 0$$

$$27z - 8 = 0$$

$$27z = 8$$

$$\underline{z = 8/27}$$

put z in (5)

$$4y - 5\left(\frac{8}{27}\right) = 0$$

$$4y - \frac{40}{27} = 0$$

$$4y = \frac{40}{27}^{10}$$

$$y = \frac{10}{27}$$

$$x = 1 - y - z$$

$$= 1 - \frac{10}{27} - \frac{8}{27}$$

$$= \frac{27 - 18}{27}$$

$$= \frac{9}{27}$$

$$x = \frac{1}{3}$$

Required stationary probability vector is $\left(1/3, \frac{10}{27}, \frac{8}{27}\right)$

(3) A habitual gambler is a member of two clubs A and B. He visits either of the clubs everyday for playing cards. He never visits

club A on two Consecutive days.
But if he visits club B on a particular day then the next day he is as likely to visit club B or club A. Find transiti-on matrix of this markov chain.
also ⓐ S.T the matrix is a regular stochastic matrix & find the unique fixed probability vector.
ⓑ If the person had visited club B on monday find the probability that he visits club A on Thursday.

Soln: The transition matrix P of markov chain is formulated as

$$P = \begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \end{array}$$

(i) $P^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$

Since all the entries of P^2 are +ve
$\therefore$ P is a regular stochastic matrix
Now we shall find unique fixed probability vector

$$VP = V$$

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$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$\begin{bmatrix} \frac{y}{2}, & x + \frac{y}{2} \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$\frac{y}{2} = x \Rightarrow 2x = y$$

$$x + y/2 = y$$

$$2x + y = 2y \Rightarrow 2x - y = 0$$

w.r.t $x + y = 1$

$$y = 1 - x$$

$$2x - 1 + x = 0$$

$$3x - 1 = 0$$

$$3x = 1 \Rightarrow x = 1/3$$

$$y = 1 - 1/3$$

$$y = \frac{2}{3}$$

Thus $V = (1/3, 2/3)$

(b) let us suppose monday as day 1 then thursday will be 3 days after monday given that the person had visited club B on monday the probability that he visits club A after 3 days is equivalent to finding $a_{21}^{(3)}$ from P^3.

$$P^3 = P^2 \cdot P = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/4 & 3/4 \\ 3/8 & 5/8 \end{bmatrix}$$

$a_{21}^{(3)} = 3/8$ required probability

* A student's study habits are as follows, if he studies one night, he is 70% sure not to study the next night, on the other hand if he does not study one night he is 60%. sure not to study the next night. In the long run how often does he study?

Soln:
A → Studying
B → not studying

$$P = \begin{array}{c|cc} & A & B \\ \hline A & 0.3 & 0.7 \\ B & 0.4 & 0.6 \end{array}$$

we have to find unique fixed probability vector $VP = V$

where $V = (x, y)$

w.r.t $x + y = 1$

$$[x \; y] \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [x \; y]$$

$$[0.3x + 0.4y \quad 0.7x + 0.6y] = [x \; y]$$

$$0.3x + 0.4y = x$$
$$0.7x + 0.6y = y$$

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$$x+y=1$$
$$y=1-x$$
$$0.3x+0.4-0.4x=x$$
$$-0.1x-x=-0.4$$
$$+1.1x=+0.4$$
$$x=\frac{0.4}{1.1}$$
$$x=0.3636$$

$$y=1-4/11$$
$$=\frac{7}{11}$$
$$\underline{y=0.6363}$$

Thus we can conclude that in the long run the student will study 4/11 of the time or 36.36 % of the time.

* The man's smoking habits are as follows. If he smokes filter cigarettes one week, he switches to non filter cigarettes the next week with probability 0.2 on the other hand if he smokes non filter cigarettes

one week there is a probability of 0.7 that he will smoke nonfilter cigarettes the next week as well. In the long run how often does he smoke filter cigarettes?

Sol^n: A: Smoking filter cigarettes
B: Smoking non filter cigarettes

State space of system $\{A, B\}$

Associated transition matrix is

$$P = \begin{array}{c c} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \end{array}$$

we have to find unique fixed probability vector $VP = V$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$\begin{bmatrix} 0.8x + 0.3y & 0.2x + 0.7y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$0.8x + 0.3y = x, \quad 0.2x + 0.7y = y$

$0.8x - x + 0.3y = 0, \quad 0.2x + 0.7y - y = 0$

$-0.2x + 0.3y = 0 \quad — ① \qquad 0.2x - 0.3y = 0 \quad — ②$

W.K.T $x + y = 1$

$y = 1 - x$

① $\Rightarrow 0.8x + 0.3 - 0.3x - x = 0$

$$-0.5x = -0.3$$

$$x = \frac{0.3}{0.5}$$

$$x = 3/5 \text{ or } 0.6$$

$$y = 1 - \frac{3}{5} = \frac{2}{5}$$

$$\text{or } y = 0.4$$

$$V = (x, y) = (3/5, 2/5) = (P_A, P_B)$$

In the long run he will smoke filter cigarettes $\frac{3}{5}$ or 60% of the time

* Three Boys A, B, C are throwing ball to each other A always throws the Ball to B & B always throws the Ball to C, C is just as likely to throw the ball to B as to A. If C was first person to throw the ball find the probability that after 3 throws.

(i) A has the ball
(ii) B has the ball
(iii) C has the ball

Soln: State space = {A B C}
t.p.m

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

Initially if C has ball

initial probability vector

$p^{(0)} = (0 \ 0 \ 1)$

Since the probability are desired after three rows we have to find

$p^{3} = p^{(0)} \cdot p^{3}$

Refer problem no. ① for p^3

$$p^{3} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$p^{(3)} = p^{(0)} p^{(3)} = (1/4, 1/4, 1/2)$

After 3 throws the probability that ball is with A is 1/4, B is 1/4 and C is 1/2.

* Two boys B_1, B_2 & two Girls G_1, G_2 are throwing ball from one to other. Each boy throws the ball to the other boy with probability 1/2 and to each girl with probability 1/4. On otherhand each girl throw the ball

to each boy with probability $\frac{1}{2}$ and never to other girl. In the long run how often does each receive the ball.

Sol^n: State space $= \{B_1, B_2, G_1, G_2\}$ and the associated t.p.m P is as follows

$$P = \begin{array}{c} \\ B_1 \\ B_2 \\ G_1 \\ G_2 \end{array}\begin{array}{c} \begin{array}{cccc} B_1 & B_2 & G_1 & G_2 \end{array} \\ \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \end{array}$$

we need to find the fixed probability vector $v = (a, b, c, d)$

i.e; $vP = v$

$$(a \ b \ c \ d)\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} = [a \ b \ c \ d]$$

$$\left[\frac{b}{2}+\frac{c}{2}+\frac{d}{2}, \ \frac{a}{2}+\frac{c}{2}+\frac{d}{2}, \ \frac{a}{4}+\frac{b}{4}, \ \frac{a}{4}+\frac{b}{4}\right] = [a \ b \ c \ d]$$

$$\frac{b}{2}+\frac{c}{2}+\frac{d}{2} = a, \quad \frac{a}{2}+\frac{c}{2}+\frac{d}{2} = b$$

$$b+c+d = 2a \quad — (1) \qquad a+c+d = 2b \quad — (2)$$

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$\frac{a}{4} + \frac{b}{4} = c, \quad \frac{a}{4} + \frac{b}{4} = d$

$a + b = 4c; - (3) \quad a + b = 4d - (4)$

W.K.T $\quad a + b + c + d = 1$

$b + c + d = 1 - a$

$|||^{ly} \quad a + c + d = 1 - b$

(1) $\Rightarrow \quad 1 - a = 2a$

$3a = 1$

$\underline{\underline{a = 1/3}}$

(2) $\Rightarrow \quad 1 - b = 2b$

$3b = 1 \Rightarrow b = 1/3$

(3) $\Rightarrow \quad 1/3 + 1/3 = 4c \Rightarrow 4c = 2/3 \Rightarrow c = 1/6$

(4) $\Rightarrow \quad 1/3 + 1/3 = 4d \Rightarrow 4d = 2/3 \Rightarrow d = 1/6$

$V = (1/3, 1/3, 1/6, 1/6)$ is the required unique fixed probability Vector

$a/4] = [a\ b\ c\ d]$